

Supersymmetry in Lorentzian Curved Spaces and Holography

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Abstract: We consider superconformal and supersymmetric field theories on four-dimensional Lorentzian curved space-times, and their five-dimensional holographic duals. As in the Euclidean signature case, preserved supersymmetry for a superconformal theory is equivalent to the existence of a charged conformal Killing spinor. Differently from the Euclidean case, we show that the existence of such spinors is equivalent to the existence of a null conformal Killing vector. For a supersymmetric field theory with an R-symmetry, this vector field is further restricted to be Killing. We demonstrate how these results agree with the existing classification of supersymmetric solutions of minimal gauged supergravity in five dimensions.

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1. Introduction

Space-time curvature can be very safely neglected in the study of the non-gravitational interactions in nature. From a theoretical point of view, however, one can expect to learn interesting lessons about quantum field theory by studying it in curved spacetimes, just like one does, for example, by varying the gauge group rank, the number and representation of fields, or sometimes even the space-time dimension. The study of supersymmetric field theories on curved spaces with Euclidean signature has recently found various applications [1–6].

When the field theories admit a holographic dual description, it is natural to study supergravity solutions comprising an asymptotically locally anti-de Sitter (AdS) space-time, where curved backgrounds arise as geometric data on the conformal boundary. For example, round d -dimensional spheres arise simply as the boundary of Euclidean AdS space in $d+1$ dimensions. If the field theory possesses an Abelian R-symmetry, then more complicated backgrounds may be obtained turning on a background gauge field coupled to the R-symmetry current, and gravity duals may be constructed in the framework of gauged supergravity. Examples in four dimensions were presented in [7,8] and involve three-dimensional Chern–Simons gauge theories in the background of certain squashed three-spheres, with non-trivial R-symmetry gauge field. Examples in five dimensions were constructed earlier in Lorentzian signature [9–11], and we will discuss them in this paper as an illustration of our general results. In one case, the Lorentzian solution can easily be Wick-rotated to Euclidean signature.

In [12,13] it was shown that four-dimensional superconformal theories (and, more generally, supersymmetric theories with an R-symmetry) remain supersymmetric in four Euclidean dimensions when the background space is *complex*. From the point of view of rigid supersymmetry, the Lorentzian signature case has so far been less studied; one exception is anti-de Sitter, which has been considered for a long time, since e.g. [14–16], and more recently in e.g. [17,18] (which also contain a more complete list of references). In this paper we consider the same question as in [12], namely under what conditions a supersymmetric field theory can preserve any supersymmetry on a curved space, in Lorentzian signature.¹ We begin by considering superconformal theories. As in [12], we find very generally that the boundary M_4 needs to admit a *conformal Killing spinor* (CKS) ϵ , possibly charged under a gauge field A_μ . The smallest amount of supersymmetry corresponds to ϵ being chiral; since the equation is linear, whenever ϵ is a conformal Killing spinor $i\epsilon$ is one too. So the minimal amount of supersymmetry is two supercharges; we focus on this case. As we will show, the condition on the geometry of M_4 for this to happen is very different from the Euclidean case. Namely, M_4 has a

¹ One might also attack this question using superspace, as in [19, Chap. 6].

conformal Killing spinor if and only if it has a *null conformal Killing vector* z . The gauge field A_μ can then be determined purely from data of the metric on M_4 .

One can also study supersymmetric theories on curved spaces using the method proposed in [20]. This consists in coupling the theory to supergravity, and then freezing its fields to background values. For a superconformal theory, the appropriate gravity theory is conformal supergravity [21–24]; we will show that the result of this procedure is again that M_4 should admit a conformal Killing spinor. For a supersymmetric theory with an R-symmetry which is not superconformal, it is natural to use new minimal supergravity [25], where the off-shell gravity multiplet contains $g_{\mu\nu}$ and two vectors a_μ , v_μ (the former coupling to the R-symmetry current). For the theory obtained by this procedure to be supersymmetric on a curved M_4 , one should then solve an equation for ϵ which is (locally) equivalent to the CKS equation, with a suitable map of a_μ , v_μ with A_μ and some data of the geometry. This map in general produces a v_μ which is complex, which in Lorentzian signature is not acceptable; imposing that it should be real turns out to require that the conformal Killing vector z is now actually a *Killing vector*. As we will see, this stronger condition arises automatically from the bulk perspective when a certain natural choice of coordinates is used.

After having determined that supersymmetry leads to clear geometrical requirements, one naturally wonders how this is related to the geometry in the bulk. The geometry of supersymmetric solutions of (Lorentzian) five-dimensional minimal gauged supergravity was considered in [9], and it is interesting to compare our result to their classification. Indeed, one of the conditions found in [9] in the bulk was the existence of a Killing vector V , which may be time-like or null. We will show that this vector always becomes null at the boundary, and reduces to the conformal Killing vector z . We will also check that the other conditions from the bulk become redundant at the boundary, in agreement with our results.

The rest of the paper is organized as follows. In Sect. 2 we show that supersymmetric asymptotically locally AdS solutions in the bulk imply the existence of a charged conformal Killing spinor on the boundary M_4 . In Sect. 3 we show that such a spinor can exist if and only if M_4 has a null conformal Killing vector, and thus that this is the condition for a superconformal theory on M_4 to preserve some supersymmetry. In Sect. 4 we extend our analysis to theories which are not necessarily superconformal, but simply supersymmetric with an R-symmetry; we show that the condition on M_4 is now that it admits a null Killing vector. In Sects. 5 and 6 we compare our results on M_4 with the bulk analysis of supersymmetric solutions of gauged minimal supergravity performed in [9], and find agreement.

2. Conformal Killing Spinors from the Bulk

In this section we discuss how supersymmetry in a holographic gravity set-up implies the existence of a conformal Killing spinor on the boundary geometry. The analysis is similar to the one performed in Euclidean signature in [12].

We consider minimal gauged supergravity in five dimensions. In Lorentzian signature $(-, +, +, +, +)$, the bosonic part of action is²

² We use notations adapted from [9], to which we refer for details. To compare with [9], one has to identify $\chi = 2\sqrt{3}\ell^{-1}$. Moreover, one needs to switch between mostly plus and mostly minus signature, which means flipping the sign of the metric and taking $\gamma_{\text{here}}^\alpha = -i\gamma_{\text{there}}^\alpha$. Also, we denote with a hat five-dimensional quantities that might be confused with four-dimensional ones.

$$S = \frac{1}{4\pi G} \int \left(\left(\frac{1}{4} \hat{R} + \frac{3}{\ell^2} \right) * 1 - \frac{1}{2} \hat{F} \wedge * \hat{F} - \frac{2}{3\sqrt{3}} \hat{F} \wedge \hat{F} \wedge \hat{A} \right), \quad (2.1)$$

where $\hat{F} = d\hat{A}$ and $\ell \neq 0$ is a real constant. We are interested in supersymmetric solutions of this theory, which are asymptotically locally AdS (with radius ℓ), and in particular we will assume the following asymptotic Fefferman–Graham form of the metric

$$\ell^{-2} d\hat{s}^2 = \frac{dr^2}{r^2} + r^2 \left(g_{\mu\nu}(x) + \mathcal{O}(r^{-1}) + \dots \right) dx^\mu dx^\nu, \quad (2.2)$$

where $\mu, \nu = 0, \dots, 3$ are curved indices and $g_{\mu\nu}(x)$ is a four-dimensional metric of Lorentzian signature $(-, +, +, +)$. The vielbein takes the form

$$\hat{e}^a = \ell r e^a + \mathcal{O}(1), \quad \hat{e}^5 = \frac{\ell dr}{r}, \quad (2.3)$$

where $a = 0, \dots, 3$ are flat four-dimensional indices and $e^a = e_\mu^a(x) dx^\mu$ is a vielbein for $g_{\mu\nu}(x)$. The associated spin connection is

$$\hat{\omega}^{ab} = \omega^{ab} + \mathcal{O}(r^{-1}), \quad \hat{\omega}^{a5} = r e^a + \mathcal{O}(r^{-1}). \quad (2.4)$$

For the bulk gauge field we assume

$$\hat{A}_\mu(x, r) = -\frac{\ell}{\sqrt{3}} A_\mu(x) + \mathcal{O}(r^{-1}), \quad \hat{A}_r(x, r) = 0, \quad (2.5)$$

which are compatible with the equations of motion. It follows that $\hat{F}_{\mu\nu} = \mathcal{O}(1)$ and $\hat{F}_{\mu r} = \mathcal{O}(r^{-2})$. The Killing spinor equation corresponding to a vanishing gravitino variation is

$$\left[\hat{\nabla}_\alpha + \frac{i}{4\sqrt{3}} (\gamma_\alpha^{\beta\gamma} - 4\delta_\alpha^\beta \gamma^\gamma) \hat{F}_{\beta\gamma} \right] \epsilon^I + \frac{1}{2\ell} \epsilon^{IJ} (i\gamma_\alpha + 2\sqrt{3}\hat{A}_\alpha) \epsilon^J = 0, \quad (2.6)$$

where we are using flat α, β five-dimensional spacetime indices. Our conventions for the spinors, which are symplectic-Majorana, can be found in Appendix A.

At leading order in the asymptotic expansion in r , the radial part of the Killing spinor equation (2.6) gives rise to

$$\partial_r \epsilon^I + \frac{i}{2r} \gamma_5 \epsilon^{IJ} \epsilon^J = 0, \quad (2.7)$$

where the index on γ_5 is flat. Note that the contribution of the gauge field strength obtained from (2.5) is sub-leading and therefore drops out. Eq. (2.7) implies that the two symplectic-Majorana spinors take the asymptotic form

$$\begin{aligned} \epsilon^1 &= r^{1/2} \epsilon + r^{-1/2} \eta + \mathcal{O}(r^{-3/2}), \\ \epsilon^2 &= i\gamma_5 (r^{1/2} \epsilon - r^{-1/2} \eta) + \mathcal{O}(r^{-3/2}), \end{aligned} \quad (2.8)$$

where ϵ and η are independent of r . Plugging these expressions back into the remaining components of (2.6), one finds that at leading order the spinors obey the following equation

$$\left(\nabla_\mu - iA_\mu \gamma_5 \right) \epsilon + \gamma_\mu \gamma_5 \eta = 0. \quad (2.9)$$

In the gamma matrix representation we adopted (see Appendix A), the symplectic-Majorana condition in five dimensions implies that the four-dimensional spinors obey $\epsilon^* = \epsilon$ and $\eta^* = -\eta$. We can also use γ_5 to define the chirality for the boundary spinors,

$$\epsilon = \epsilon_+ + \epsilon_-, \quad \eta = \eta_+ + \eta_-, \quad (2.10)$$

where $\gamma_5 \epsilon_{\pm} = \pm \epsilon_{\pm}$ and $\gamma_5 \eta_{\pm} = \pm \eta_{\pm}$. Taking the trace of (2.9) allows us to solve for η :

$$\eta = -\frac{1}{4}(\gamma_5 \nabla_{\mu} + i A_{\mu}) \gamma^{\mu} \epsilon. \quad (2.11)$$

Finally, inserting this back into (2.9), we find

$$\nabla_{\mu}^A \epsilon_+ = \frac{1}{4} \gamma_{\mu} D^A \epsilon_+, \quad (2.12)$$

where $\nabla_{\mu}^A = \nabla_{\mu} - i A_{\mu}$ and $D^A = \gamma^{\mu} \nabla_{\mu}^A$. This is the equation for a charged conformal Killing spinor and will be the starting point of our subsequent analysis. Note that a similar equation is given for ϵ_- by complex conjugation.

2.1. Conformal Killing spinors and superconformal theories. The Eq. (2.12) was derived using holographic methods but its use is not limited to theories with an holographic dual. Indeed, the existence of a charged conformal Killing spinor is precisely the condition that allows one to preserve supersymmetry for any superconformal field theory on a curved background. This has been discussed in detail in [12] for the Euclidean case. It works similarly in Lorentzian signature and we will now review the argument.

In order to define a supersymmetric theory on a curved manifold M we can use the strategy of [20] which consists in coupling the theory to supergravity and then freeze the fields of the gravitational multiplet. The value of the auxiliary fields determines the coupling of the theory to the curved background.

The appropriate supergravity for a superconformal theory is conformal supergravity, whose fields are $g_{\mu\nu}$, ψ_{μ} and A_{μ} . In order to preserve some supersymmetry, the gravitino variation must vanish. With obvious redefinitions it reads [23, 26]

$$\delta \psi_{\mu} = (\nabla_{\mu} - i A_{\mu} \gamma_5) \epsilon + \gamma_{\mu} \gamma_5 \eta, \quad (2.13)$$

where ϵ is the parameter for the supersymmetries Q and η for the superconformal transformations S . We see that the vanishing of (2.13) is the same as Eq. (2.9) which, in turn, is equivalent to the CKS equation.

It is crucial for the argument that the algebra of the superconformal transformations of $g_{\mu\nu}$, ψ_{μ} , A_{μ} closes off shell [27]. Therefore the variation (2.13) depends only on the background field A_{μ} and is not modified by the coupling to matter. Moreover, the supergravity action for the fields $g_{\mu\nu}$, ψ_{μ} , A_{μ} is separately invariant and can be safely omitted without spoiling the superconformal invariance of the matter part.

In the next section we will discuss what the existence of a conformal Killing spinor implies for the geometry of the four-dimensional space-time.

3. Geometry of Conformal Killing Spinors in Lorentzian Signature

In this section we will analyse the geometrical content of conformal Killing spinors (2.12), charged under a gauge field A . This equation is also known as *twistor equation*, and it is well-studied in conformally flat spaces [28]. The case where $A = 0$ has already been analysed in [29]: all possible spaces on which a conformal Killing spinor exists were classified. It turns out that they fall in two classes: Fefferman metrics, and pp-wave spacetimes. We will review these two as particular cases (with $A = 0$) of our more general classification in Sect. 4.4. As stated in the introduction, we will find that a charged conformal Killing spinor exists if and only if there exists a null conformal Killing vector. To explain the computations that lead to this result, we need to review first some geometrical aspects of four-dimensional spinors in Lorentzian signature.

3.1. Geometry defined by a spinor. In this section we review the geometry associated with a Weyl³ spinor ϵ_+ in Lorentzian signature. As in Sect. 2, we work in the signature $(-, +, +, +)$ and with real gamma matrices. We start with a spinor of positive chirality ϵ_+ and its complex conjugate $\bar{\epsilon}_- \equiv (\epsilon_+)^*$. We can use ϵ_+ and $\gamma_\mu \epsilon_-$ to form a basis for the spinor of positive chirality and $\bar{\epsilon}_-$ and $\gamma_\mu \bar{\epsilon}_+$ for those of negative chirality. A convenient way of choosing the basis is obtained as follows. At every point where ϵ_+ is not vanishing, it defines a real null vector z and a complex two form ω . We can express this fact in terms of bispinors⁴

$$\epsilon_+ \otimes \bar{\epsilon}_+ = z + i * z, \quad \epsilon_+ \otimes \bar{\epsilon}_- \equiv \omega, \quad (3.1)$$

where, as usual, $\bar{\epsilon} = \epsilon^\dagger \gamma^0$. Equivalently, as spinor bilinears the forms read

$$z_\mu = \frac{1}{4} \bar{\epsilon}_+ \gamma_\mu \epsilon_+, \quad \omega_{\mu\nu} = -\frac{1}{4} \bar{\epsilon}_- \gamma_{\mu\nu} \epsilon_+. \quad (3.2)$$

It can be shown easily that $z \wedge \omega = 0$, which implies that we can write

$$\omega = z \wedge w \quad (3.3)$$

for some complex one-form w . The form ω looks very similar to the holomorphic top-form of an almost complex structure; in Sect. 4.2 we will make this similarity more precise by introducing the concept of *CR-structure*. One can then show that the spinor ϵ_+ is annihilated by z and w , namely⁵

$$z \cdot \epsilon_+ = w \cdot \epsilon_+ = 0, \quad (3.4)$$

and

$$z^2 = z \cdot w = w^2 = 0, \quad w \cdot \bar{w} = 2. \quad (3.5)$$

We can think of z and w as elements of a local frame: $z = e^+$, $w = e_2 - i e_3$. In order to complete the frame we can introduce another real one-form e^- such that

$$(e^-)^2 = 0, \quad e^- \cdot z = 2, \quad e^- \cdot w = 0. \quad (3.6)$$

³ We could also use a Majorana spinor.

⁴ We are using conventions where $*\alpha \wedge \alpha = ||\alpha||^2 \text{Vol}_4$ and $\text{Vol}_4 = e^{0123} = \frac{1}{2} e^{+-23}$.

⁵ When acting on a spinor the dot denotes Clifford multiplication, as in $z \cdot \epsilon = z^\mu \gamma_\mu \epsilon$.

The four-dimensional metric then takes the form

$$ds^2 = z e^- + w \bar{w}. \quad (3.7)$$

Notice that the pair (z, ω) is uniquely defined by the spinor, while the frame $\{z, e^-, w, \bar{w}\}$ is not. This is because, given a w that satisfies (3.3), any other one-form of the form $w + \alpha z$ still satisfies it. After having fixed w , e^- is uniquely determined by the conditions (3.6). Alternatively, one can pick any null e^- such that $e^- \cdot z = 2$; a complex w orthogonal to e^- and z and such that $w^2 = 0$, $w \cdot \bar{w} = 2$ is then uniquely determined.

In summary, the vielbein $\{z, e^-, w, \bar{w}\}$ is not uniquely determined by ϵ_+ ; rather, it is determined up to the ambiguity

$$w \rightarrow w + \alpha z, \quad e^- \rightarrow e^- - \bar{\alpha} w - \alpha \bar{w} - |\alpha|^2 z. \quad (3.8)$$

The complex function α has to do with the fact that ϵ_+ by itself describes an \mathbb{R}^2 structure,⁶ rather than the identity structure that would be described by the vielbein $\{z, e^-, w, \bar{w}\}$.

3.2. Intrinsic torsions. We are now ready to define a basis of spinors. For the positive chirality we can take

$$\epsilon_+, \quad e^- \cdot \epsilon_- \quad (3.9)$$

and for negative chirality

$$\epsilon_-, \quad e^- \cdot \epsilon_+. \quad (3.10)$$

It follows from (3.4) and the previous definitions that

$$\gamma^\mu \epsilon_+ = -w^\mu \epsilon_- + \frac{1}{2} z^\mu e^- \cdot \epsilon_+. \quad (3.11)$$

Using the basis (3.9), we can expand

$$\nabla_\mu \epsilon_+ = p_\mu \epsilon_+ + q_\mu e^- \cdot \epsilon_-. \quad (3.12)$$

p_μ, q_μ are (locally) complex one-forms. They can be interpreted as intrinsic torsions for the \mathbb{R}^2 structure defined by ϵ_+ .⁷ It is also possible to express p and q in terms of exterior differentials. In order to do so, we can use the auxiliary piece of data e^- , which allows to define a vielbein $\{z, e^-, w, \bar{w}\}$, as described in Sect. 3.1. This vielbein is an identity structure. The intrinsic torsion of an identity structure e^a is expressed by the “anholonomy coefficients” c^a_{bc} defined by $de^a = c^a_{bc} e^b \wedge e^c$. As shown in Appendix B, we can parametrize the de^a as

$$dz = 2\text{Re } p \wedge z + 4\text{Re}(q \wedge \bar{w}), \quad (3.13a)$$

$$dw = -2\rho \wedge z + 2i\text{Im } p \wedge w - 2q \wedge e^-, \quad (3.13b)$$

$$de^- = 4\text{Re}(\rho \wedge \bar{w}) - 2\text{Re } p \wedge e^-. \quad (3.13c)$$

Here p and q are precisely the one-forms appearing in the covariant derivative of the spinor (3.12), while ρ is a new one-form which is an intrinsic torsion for the identity

⁶ The stabilizer of the light-like vector z is $\text{SO}(2) \ltimes \mathbb{R}^2$; w breaks the $\text{SO}(2)$ to the identity. For more details see [30]. See also section 4 of [31] for a similar discussion in six dimensions.

⁷ For a G -structure, one decomposes $\Lambda^2 T = g \oplus k$ (where g is the Lie algebra of G); the intrinsic torsion is then given by $k \otimes T$. In our case, $G = \mathbb{R}^2$, so k is 4-dimensional, and $k \otimes T$ is 16-dimensional. These are precisely the eight complex components of the complex one-forms p and q .

structure $\{z, e^-, w, \bar{w}\}$ but not for the \mathbb{R}^2 structure defined by ϵ_+ . In four dimensions, we have $4 \times 6 = 24$ real anholonomy coefficients, which we can identify with three complex one-forms p, q and ρ .

Alternatively, we can extract p and q from the forms z and ω defined in (3.1). One can indeed derive the following differential constraints

$$dz = 2\text{Re } p \wedge z + 4\text{Re}(q \wedge \bar{w}), \quad (3.14a)$$

$$d\omega = 2p \wedge \omega - 2q \wedge (z \wedge e^- + w \wedge \bar{w}), \quad (3.14b)$$

$$(e^- \cdot \nabla) \omega = 2(p \cdot e^-) \omega - 2(q \cdot e^-)(z \wedge e^- + w \wedge \bar{w}), \quad (3.14c)$$

which allow to determine p and q from the geometry. Notice that dz and $d\omega$ alone would not be enough to determine p and q .

3.3. Conformal Killing spinors are equivalent to conformal Killing vectors. In this section we study the geometrical constraints imposed by the existence of a charged conformal Killing spinor ϵ_+ . As discussed in Sect. 2.1, on the resulting curved backgrounds one can define a superconformal field theory preserving some supersymmetry. We will show that existence of a charged conformal Killing spinor is equivalent to the existence of a conformal Killing vector. In turn, this allows to introduce local coordinates, in which the metric (or equivalently the frame) takes a canonical form, generalizing the one discussed in [29], corresponding to $A = 0$. We shall present this metric in Sect. 4, in the case when the conformal Killing vector becomes a Killing vector. The two metrics are simply related by a Weyl rescaling.

Similarly to what is known for uncharged conformal Killing spinors [32], it is straightforward to show that if ϵ_+ is a charged conformal Killing spinor, the null vector z_μ defined in (3.2) is a conformal Killing vector. We now show that also the opposite is true.

First of all, notice that not only does a spinor ϵ_+ determine a null vector z (via (3.1) or (3.2)), but also that in a sense the opposite is true. Indeed, let us study the map $\epsilon_+ \mapsto z$. The space of spinors with fixed $\epsilon_+^t \epsilon_+$ is an S^3 in the four-dimensional space of all spinors. This is mapped by (3.2) into the space of all null vectors z with fixed z^0 , which is an S^2 (the so-called “celestial sphere”). This is the Hopf fibration map, whose fibre is an S^1 . So, to any null vector z one can associate a $U(1)$ worth of possible spinors ϵ_+ whose bilinear is z .

Let us now move on to differential constraints. We consider the equation defining a charged conformal Killing spinor, or twistor-spinor,

$$\nabla_\mu^A \epsilon_+ = \frac{1}{4} \gamma_\mu D^A \epsilon_+, \quad (3.15)$$

where $\nabla_\mu^A = \nabla_\mu - i A_\mu$ and D^A is the covariantized Dirac operator $D^A = \gamma^\mu \nabla_\mu^A$. Note that the equation does not mix chiralities, and we consider the case of a positive chiral spinor. A is a real connection and ϵ_+ is a section of the $U(1)$ Hopf fibration described in the previous paragraph.

We can expand Eq. (3.15) in the basis (3.9). Using (3.12), we obtain a set of linear equations for p, q and the gauge field A . Since the gamma-trace of equation (3.15) is trivial we find a total of six complex constraints

$$\begin{aligned} q \cdot z &= 0, & p^A \cdot e^- &= 0, & p^A \cdot z &= 2q \cdot \bar{w}, \\ q \cdot w &= 0, & p^A \cdot \bar{w} &= 0, & p^A \cdot w &= -2q \cdot e^-, \end{aligned} \quad (3.16)$$

where $p_\mu^A = p_\mu - i A_\mu$.

Two of these conditions will determine the real gauge field A . The remaining eight real conditions are constraints to be imposed on the geometry. We now show that these constraints are equivalent to the existence of a conformal Killing vector.

A short computation shows that

$$\nabla_\mu z_\nu = 2\text{Re} \left(p_\mu z_\nu + 2\bar{q}_\mu w_\nu \right). \quad (3.17)$$

Taking the anti-symmetric part of this equation we reproduce the first equation in (3.13). Taking the symmetric part and imposing that z is a conformal Killing vector,

$$(\mathcal{L}_z g)_{\mu\nu} = 2\nabla_{(\mu} z_{\nu)} = \lambda g_{\mu\nu}, \quad (3.18)$$

we obtain the conditions

$$\begin{aligned} \text{Re } p \cdot e^- &= 0, \quad q \cdot z = q \cdot w = 0, \\ \text{Re } p \cdot z &= 2\text{Re}(q \cdot \bar{w}), \quad \text{Re } p \cdot w = -q \cdot e^-, \end{aligned} \quad (3.19)$$

with

$$\lambda \equiv 4\text{Re}(q \cdot \bar{w}). \quad (3.20)$$

This set of eight real conditions is precisely the subset of the constraints (3.16) not involving A , as previously stated.

To summarise, we showed that on any manifold M_4 with a null conformal Killing vector we can find a charged conformal Killing spinor. In Sect. 3.5 we will give an expression for the gauge field A under which the conformal Killing spinor is charged.

Notice that the presence of a conformal Killing spinor also implies

$$d\omega = i \left(2A - 3 * (q \wedge e^- \wedge \bar{w}) \right) \wedge \omega. \quad (3.21)$$

In the Euclidean signature case, one finds [12] a very similar condition, $d\omega = W \wedge \omega$, where ω is the $(2, 0)$ form of a complex structure. While in that case that condition turns out to be necessary and sufficient for the existence of a charged conformal Killing spinor, in the present case of Lorentzian signature this condition alone is not sufficient to imply supersymmetry.

3.4. Conformal Killing spinors are equivalent to conformal Killing–Yano forms. Our two-form ω satisfies an interesting property, namely it is a *charged* conformal Killing–Yano form (CKF). In general, a p -form φ on a d -dimensional space(-time) (M, g) is conformal Killing–Yano (or simply conformal Killing) if it satisfies the equation

$$\nabla_\rho \varphi_{\mu_1 \dots \mu_p} = \nabla_{[\rho} \varphi_{\mu_1 \dots \mu_p]} + \frac{p}{d-p+1} g_{\rho[\mu_1} \nabla^\sigma \varphi_{|\sigma| \mu_2 \dots \mu_p]}. \quad (3.22)$$

This is a conformally invariant equation: if φ is a conformal Killing form on (M, g) and the metric is rescaled as $g \rightarrow \tilde{g} = e^{2f} g$, then the rescaled form $\tilde{\varphi} = e^{(p+1)f} \varphi$ is conformal Killing on (M, \tilde{g}) .

In the uncharged case ($A = 0$), it is known that the bilinears of conformal Killing spinors are conformal Killing forms, see e.g. [33]. We already saw that this is true for z , since a conformal Killing one-form is just the dual of a conformal Killing vector. For a two-form in four dimensions, this is easiest to check in the two-component formalism for spinors. Because of its definition in (3.1), ω can be written as $\omega^{\alpha\beta} = \epsilon^\alpha \epsilon^\beta$, and the CKF equation (3.22) reads

$$D_{\dot{\alpha}}^{(\beta} \omega^{\gamma\delta)} = 0. \quad (3.23)$$

The CKS equation reads in this formalism $D_{\dot{\alpha}}^{(\beta} \epsilon^{\gamma)} = 0$, which implies obviously (3.23). Since for us the CKS is actually charged under A , we obtain that $(\nabla^A)_{\dot{\alpha}}^{(\beta} \omega^{\gamma\delta)} = 0$, where $\nabla_{\mu}^A = \nabla_{\mu} - 2iA_{\mu}$; or, going back to four-component language,

$$\nabla_{\rho}^A \omega_{\mu\nu} = \nabla_{[\rho}^A \omega_{\mu\nu]} - \frac{2}{3} g_{\rho[\mu} \nabla^A{}^{\sigma} \omega_{\nu]\sigma}, \quad (3.24)$$

which is a charged version of the standard conformal Killing form equation.

It is interesting to ask to what extent this property can be used to characterize our spacetime, similarly to what we saw in Sect. 3.3. First of all, we should ask when a two-form ω can be written as a spinor bilinear as in (3.1). One possible answer is that the form should define an \mathbb{R}^2 structure; namely, that the stabilizer of ω in $\text{SO}(3,1)$ should be \mathbb{R}^2 . We can also give an alternative, more concrete characterization by using again the two-component formalism for spinors. The two-form ω should be imaginary self-dual, which means it is in the $(1,0)$ representation of $\text{SO}(3,1)$; the corresponding bispinor then is a symmetric matrix $\omega^{\alpha\beta}$. As a 2×2 matrix, this can be factorized as $\epsilon^{\alpha} \epsilon^{\beta}$ if and only if it has rank 1, which is equivalent to $\det(\omega) = \frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} \omega^{\alpha\gamma} \omega^{\beta\delta} = 0$; in the original form language, $\omega_{\mu\nu} \omega^{\mu\nu} = 0$. So we have obtained that a two-form ω can be written as a bispinor as in (3.1) if and only if it is imaginary self-dual and null:

$$\omega = \epsilon_+ \otimes \overline{\epsilon_-} \iff \begin{cases} *\omega = i\omega, \\ \omega^{\mu\nu} \omega_{\mu\nu} = 0. \end{cases} \quad (3.25)$$

Remarkably, it turns out that the content of the Eq. (3.24) for a CKF is exactly the same as the content of the system (3.15) for a CKS. Indeed, if one uses (3.12) in (3.24) (or in (3.23)), the system one finds is exactly (3.16). Thus, we can conclude that a choice of metric and gauge field A admits a charged CKS if and only if it admits a null, imaginary self-dual charged CKF.

This reformulation is slightly less interesting than the one involving a null CKV in Sect. 3.3. Although CKF's do have physical applications (such as helping in finding first integrals of the geodesic equation, see for example [34]), their geometrical meaning is less compelling than that of a CKV. Moreover, one needs both the data of the geometry and of the gauge field A to check the condition (3.24), whereas in the previous section we saw that the presence of a null CKV tells us that a geometry can admit a charged CKS for some A (without having to guess its form, which will actually be determined in Sect. 3.5). Last but not least, the CKV condition is computationally easier to check than the CKF condition (3.24).

3.5. Determining the gauge field. The gauge field A can be determined by the four equations in (3.16) involving A . One possible expression is

$$A = \text{Im}(p + i * (q \wedge e^- \wedge \bar{w})). \quad (3.26)$$

Here, p and q are intrinsic torsion forms that can be computed for example from (3.13). The fact that (3.26) involves e^- might look puzzling, since, as we stressed in Sect. 3.1, e^- is an auxiliary degree of freedom, not one determined by the spinor ϵ_+ . More precisely, the vielbein $\{z, e^-, w, \bar{w}\}$ is only defined up to the freedom (3.8). From the definition of p and q in (3.12), using (3.11) we see that under (3.8)

$$p_{\mu} \rightarrow p_{\mu} - 2\bar{\alpha} q_{\mu}, \quad q_{\mu} \rightarrow q_{\mu}. \quad (3.27)$$

Using this and (3.26), we can show that A is invariant under (3.8). This means it is independent on the choice of e^- , and is in fact determined by ϵ_+ alone.

We now show that the gauge curvature is invariant under the action of the vector field z , namely that

$$\mathcal{L}_z F = 0. \quad (3.28)$$

We first compute the Lie derivative of a set of vielbein with respect to the vector z . Using Eq. (3.13) and the constraints (3.16) imposed by the conformal Killing spinor equation we find

$$\begin{aligned} \mathcal{L}_z z &= \iota_z dz = \lambda z, \\ \mathcal{L}_z w &= \iota_z dw = -2(\text{Re } p \cdot w + \rho \cdot z)z + \left(\frac{\lambda}{2} + iz \cdot A + 3i\text{Im}(q \cdot \bar{w}) \right) w, \\ \mathcal{L}_z e^- &= \iota_z de^- = 2(\text{Re } p \cdot \bar{w} + \bar{\rho} \cdot z)w + 2(\text{Re } p \cdot w + \rho \cdot z)\bar{w}, \end{aligned} \quad (3.29)$$

where λ has been defined in (3.20). In order to simplify these expressions we can make use of the freedom in the choice of a basis (3.8) to set

$$\text{Re } p \cdot \bar{w} + \bar{\rho} \cdot z = 0 \quad (3.30)$$

and the gauge invariance to impose

$$z \cdot A = -3 \text{Im}(q \cdot \bar{w}). \quad (3.31)$$

At this point, the Lie derivative of the vielbein simply becomes⁸

$$\mathcal{L}_z z = \lambda z, \quad \mathcal{L}_z w = \frac{\lambda}{2} w, \quad \mathcal{L}_z e^- = 0, \quad (3.32)$$

which is consistent with (3.18). We can also take the Lie derivative of (3.13) to compute the Lie derivatives of the torsions

$$\begin{aligned} \mathcal{L}_z p &= \frac{1}{4}(d\lambda \cdot z)e^- + \frac{1}{4}(d\lambda \cdot w)\bar{w}, \\ \mathcal{L}_z q &= \frac{\lambda}{2}q - \frac{1}{8}(d\lambda \cdot z)w - \frac{1}{8}(d\lambda \cdot w)z, \\ \mathcal{L}_z \rho &= -\frac{\lambda}{2}\rho - \frac{1}{8}(d\lambda \cdot w)e^- + \frac{1}{8}(d\lambda \cdot e^-)w. \end{aligned} \quad (3.33)$$

It is then straightforward to check from Eq. (3.26) that in our gauge $\mathcal{L}_z A = 0$. It follows that

$$\iota_z F = \mathcal{L}_z A - d(z \cdot A) = d(3\text{Im}(q \cdot \bar{w})). \quad (3.34)$$

Notice that this expression is independent of the choice of gauge and frame (due to (3.16)) and it is valid in general. It follows from (3.34) that F is invariant, $\mathcal{L}_z F = 0$.

4. Supersymmetric Theories with an R-Symmetry

In this section we will discuss an alternative supersymmetry equation, that arises as the rigid limit of new minimal supergravity [25,35]. This formulation is particularly well suited to describe supersymmetric field theories with an Abelian R-symmetry, and it may be thought as a special case of the CKS equation.

⁸ As we will see in Sect. 4, in the new minimal case $\lambda = 0$, hence z is a Killing vector and the Lie derivative of the vielbein vanishes, $\mathcal{L}_z z = \mathcal{L}_z w = \mathcal{L}_z e^- = 0$. In the notation of Sect. 4 the gauge condition (3.31) reads $a \cdot z = 0$.

4.1. New minimal supersymmetry equation. Solutions of the conformal Killing spinor equation (3.15) are closely related to solutions of the supersymmetry equation

$$\nabla_\mu \epsilon_+ = -i \left(\frac{1}{2} v^\nu \gamma_{\nu\mu} + (v - a)_\mu \right) \epsilon_+, \quad (4.1)$$

arising from the rigid limit of new minimal supergravity⁹ [25, 35]. Here a and v are real vectors and v is required to satisfy $d * v = 0$. When this condition has a solution, we can consistently define supersymmetric field theories on the four-manifold M_4 , with background fields v and a , using the strategy of [20].

It is simple to see that a solution of (4.1) is a conformal Killing spinor associated with the gauge field $A = a - \frac{3}{2}v$. It follows from our analysis in Sect. 3.3 that there should exist a null conformal Killing vector. It is in fact straightforward to see with a direct computation that Eq. (4.1) implies that $z_\mu = \frac{1}{4}\bar{\epsilon}_+ \gamma_\mu \epsilon_+$ is not only conformal Killing, but actually even Killing.

Vice versa, if we start with a solution of the conformal Killing spinor equation (3.15) without zeros, charged under a connection A , we can define a complex vector v through

$$D^A \epsilon_+ \equiv 2i v \cdot \epsilon_+. \quad (4.2)$$

Every spinor of negative chirality can indeed be written as a linear combination of gamma matrices acting on ϵ_+ . If ϵ_+ has no zeros, v is defined everywhere.¹⁰ Using (3.12) we can express some components of v in terms of q

$$w \cdot v = 2i q \cdot e^-, \quad z \cdot v = -2i q \cdot \bar{w}. \quad (4.3)$$

All other components of v are immaterial and v itself is not uniquely determined, since we can always add to it a term along z and w (recall that $z \cdot \epsilon_+ = w \cdot \epsilon_+ = 0$). We can use this freedom to make v real, except for an imaginary part given by

$$\text{Im } v = -\frac{\lambda}{4} e^-, \quad (4.4)$$

where λ was defined in (3.20). This rewriting of q in terms of v will be useful in Sect. 5, where we will perform a comparison between bulk and boundary solutions. It is now easy to show that (3.15) can be rewritten as Eq. (4.1) with $a = A + \frac{3}{2}v$.

So far, all we have done is rewriting the equation for conformal Killing spinors (3.15) as (4.1); v , however, is potentially still complex, with imaginary part given by (4.4), and in general $d * v \neq 0$. We will now show that v can be made real by an appropriate Weyl rescaling $g_{\mu\nu} \rightarrow e^{2f} g_{\mu\nu}$. To see this, remember that z is a conformal Killing vector, namely a vector satisfying (3.18). However, a null conformal Killing vector can always be made a null Killing vector by a Weyl transformation. In particular,

$$\mathcal{L}_z g_{\mu\nu} = \lambda g_{\mu\nu} \Rightarrow \mathcal{L}_z (e^{2f} g_{\mu\nu}) = (\lambda + 2z \cdot df) g_{\mu\nu}. \quad (4.5)$$

In coordinates where $z = \frac{\partial}{\partial y}$, it is then enough to solve $2 \frac{\partial f}{\partial y} = -\lambda$. This is possible as long as there are no closed time-like curves. In the rescaled metric $e^{2f} g_{\mu\nu}$, z is now a Killing vector, which implies that $\lambda = 0$; from (4.4), we then see that v is real.

⁹ We use lower-case letters a and v for the auxiliary fields of new minimal supergravity, in order to avoid confusion with the A of conformal supergravity we have been using until now.

¹⁰ Conformal Killing spinors with zeros do exist; see for example [36] for a characterization.

Moreover, a similar argument shows that we can use the remaining ambiguity in shifting the z component of v to arrange for

$$d * v = 0. \quad (4.6)$$

Hence we have shown that, by a conformal rescaling of the metric, one can take the charged conformal Killing spinor equation (3.15) to the condition of unbroken supersymmetry in new minimal supergravity (4.1). The fact that one can bring (3.15) to (4.1) was to be expected because of the formalism of conformal compensators (for a review see [26]). In that formalism, one obtains new minimal supergravity by coupling a tensor multiplet to conformal supergravity, and by then giving an expectation value to the tensor multiplet.

To summarize, the geometrical constraints imposed by the new minimal equation just amounts to the existence of a null Killing vector z . As a check, we can count components. The new minimal equation (4.1) brings 16 real constraints and the existence of a Killing vector brings 9 real conditions. The remaining 7 real constraints can be used to determine the components of the gauge fields: 4 for a and 3 for v . a and v can now be computed as follows. v can be computed from (4.3), and from (4.6), while $a = A + \frac{3}{2}v$, where A was given in (3.26). Recall that the intrinsic torsion p and q can be computed for example from (3.13).

Finally, we observe that a solution ϵ_+ of the new minimal equations is defined up to a multiplication by a complex number. We can form two independent Majorana spinors $\epsilon_1 = \epsilon_+ + \epsilon_-$ and $\epsilon_2 = i(\epsilon_+ - \epsilon_-)$ corresponding to two independent real supercharges. The commutator of these two supersymmetries closes on the isometry generated by z :

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \Phi = \mathcal{L}_z^a \Phi, \quad (4.7)$$

where Φ is any field in the Lagrangian and the Lie derivative \mathcal{L}^a is covariantized with respect to a . The superalgebra can be easily extracted from the transformation rules of matter fields in the new minimal supergravity or from the algebra of local supergravity transformations [25, 35].

4.2. Introducing coordinates. We can obtain more explicit expressions for v and A after introducing a local set of coordinates, as promised earlier. First, notice that, using (4.3), (4.4) and the fact that $\lambda = 0$, Eqs. (3.13a) and (3.21) simplify considerably:

$$dz = -2\iota_v * z, \quad (4.8a)$$

$$d\omega = 2ia \wedge \omega. \quad (4.8b)$$

As noticed at the end of Sect. 3.3, the second of these equations is similar to the equation that in Euclidean signature implies that the manifold is complex. (4.8) can be used to compute all the components of a and v not along z . In particular, (4.8a) can be inverted to give

$$v^\perp \equiv v - \frac{1}{2}(e^- \cdot v)z = -\frac{1}{4}\iota_{e^-} * dz. \quad (4.9)$$

As discussed before, the component of v along z is ambiguous and is determined by requiring (4.6).

Given a null Killing vector, there exists a set of natural coordinates adapted to this. We will follow the discussion in [31]. We can introduce a coordinate y such that as vector field

$$z = \frac{\partial}{\partial y}, \quad (4.10)$$

and then the vector field dual to the one-form e^- introduced earlier can be parameterized as

$$e^- = 2H \left(\frac{\partial}{\partial u} - \mathcal{F} \frac{\partial}{\partial y} \right), \quad (4.11)$$

for some H and \mathcal{F} . Taking as coordinates on the four-dimensional space (y, u, x^m) , the functions H and \mathcal{F} do not depend on y , and are otherwise arbitrary functions of u and x^m . In these coordinates, the four-dimensional metric can be written as

$$ds^2 = 2H^{-1}(du + \beta)(dy + \varrho + \mathcal{F}(du + \beta)) + Hh_{mn}dx^m dx^n, \quad (4.12)$$

where h_{mn} is a two-dimensional metric, while $\beta = \beta_m dx^m$ and $\varrho = \varrho_m dx^m$ are one-forms. Everything depends on u and x^m , but not on y . Therefore, as one-forms,

$$z = H^{-1}(du + \beta), \quad e^- = 2(dy + \varrho + \mathcal{F}Hz). \quad (4.13)$$

The remaining elements of the vielbein can be complexified as $w = e^2 - ie^3$.

Our four-dimensional manifold M_4 can be seen as an \mathbb{R} fibration (with coordinate y) over a three-dimensional manifold M_3 (spanned by $\{z, w\}$). The latter admits a *CR structure*: namely, a one-dimensional complex subbundle $T_{1,0} \subset TM_3$, such that $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$. Roughly speaking, this can be thought of as a complex structure on two of the three dimensions of M_3 . For us, $T_{1,0}$ is spanned by the vector dual to the one-form \bar{w} . From a dual point of view, the subbundle of T^*M_3 spanned by one-forms which are orthogonal to $\bar{T}_{1,0}$ has dimension two, and it is spanned by the one-forms we have been calling z and w ; so its volume form is $z \wedge w$, which is the form we have been calling ω , and which in a sense can be used to characterize the CR structure. The role of $z \wedge w$ actually becomes clearer in higher odd dimension $2n+1$; the bundle $T_{1,0}$ is now an n -dimensional bundle which should be closed under Lie bracket (just like for a complex manifold). The subbundle of T^*M_3 orthogonal to $\bar{T}_{1,0}$ now has dimension $n+1$, and it is spanned by forms z, w_1, \dots, w_n . Integrability of $T_{1,0}$ is equivalent to the statement that $d(z \wedge w_1 \wedge \dots \wedge w_n) = a \wedge (z \wedge w_1 \wedge \dots \wedge w_n)$ for some one-form a . Summing up, on our three-dimensional manifold M_3 the form $z \wedge w$ is the analogue of a holomorphic volume form for a CR-structure, and can be used to characterize it.

Let us now present expressions for v and A in these coordinates. Evaluating (4.9) we find

$$v^\perp = \frac{1}{4}H^{-2}[*_2(\beta \wedge \partial_u \beta - d_2 \beta)]e^- + \frac{1}{2}H *_2 \left[\partial_u(H^{-1}\beta) - d_2(H^{-1}) \right], \quad (4.14)$$

where we defined $d_2 = dx^m \partial_m$ and $*_2$ is the Hodge star operator with respect to the metric h_{mn} . Inserting $a \equiv a^\perp + \frac{1}{2}(a \cdot e^-)z$ into (4.8b), we determine a^\perp as

$$a^\perp = \frac{1}{4} *_2 \left[d_2(H^{-1}\bar{w}) - \partial_u(H^{-1}\beta \wedge \bar{w}) \right] w + \text{c.c.}, \quad (4.15)$$

where c.c. denotes the complex conjugate.¹¹ The remaining component of the gauge field is given by $a \cdot e^- = A \cdot e^- + \frac{3}{2}v \cdot e^-$. As already noticed the component $v \cdot e^-$ is

¹¹ Comparing with Sect. 3.5 we see that $a \cdot z = 0$ is a consequence of the Lie derivative \mathcal{L}_z of our vielbein being zero, and corresponds to the gauge condition (3.31) (see footnote 8).

ambiguous; $A \cdot e^-$ can be extracted for example from the second and third equations in (B.7) and reads

$$A \cdot e^- = \frac{1}{2} H^{-1} *_2 [d_2 \varrho + \mathcal{F} d_2 \beta + (\partial_u \varrho + \mathcal{F} \partial_u \beta) \wedge \beta + H \operatorname{Re}(\bar{w} \wedge \partial_u w)]. \quad (4.16)$$

Perhaps it is worth emphasizing that the data entering in the metric (4.12), gauge field A and v ($H, \mathcal{F}, \beta, \varrho, h_{mn}$) are completely arbitrary. This is in stark contrast with the typical situation in supergravity, where e.g. the Bianchi identities and equations of motion impose more stringent constraints on the geometry.

4.3. Non-twisting geometries. In the special case that $z \wedge dz = 0$ everywhere, z is hypersurface orthogonal, in the sense that the distribution defined by vectors orthogonal to z is integrable (by Frobenius theorem). As we will show in Sect. 6, this corresponds to the case where the Killing vector in the bulk is *null*. Since z is hypersurface orthogonal, there exist preferred functions H and u such that

$$z = H^{-1} du. \quad (4.17)$$

Comparing with Eq. (4.13), we see that in these particular coordinates $\beta = 0$. After performing a further local change of coordinates to eliminate ϱ , the metric can be brought to the pp-wave form, namely

$$ds^2 = 2H^{-1} du (dy + \mathcal{F} du) + H h_{mn} dx^m dx^n. \quad (4.18)$$

In addition we have

$$v^\perp = -\frac{1}{2} H *_2 d_2(H^{-1}) \quad (4.19)$$

$$a^\perp = \frac{1}{4} \left[*_2 d_2(H^{-1} \bar{w}) \right] w + \text{c.c.} \quad (4.20)$$

$$A \cdot e^- = \frac{1}{2} *_2 [\operatorname{Re}(\bar{w} \wedge \partial_u w)] \quad (4.21)$$

where in particular notice $v \cdot z = 0$.

4.4. The case $A = 0$. It is interesting to study what happens in the particular case $A = 0$. Actually, as we showed in this section, every solution to the new minimal equation (4.1) is also a solution to the CKS equation (3.15), and hence must be included in the classification of uncharged conformal Killing spinors obtained in [29]. We will first consider the case $z \wedge dz \neq 0$, and then the case $z \wedge dz = 0$.

When $z \wedge dz \neq 0$, z is a *contact form* on the three-dimensional manifold M_3 , spanned by $\{z, w\}$. It follows from (4.8a) that $\frac{1}{2} z \cdot v \equiv v_- \neq 0$; using (3.8), we can then make $w \cdot v = 0$, so that¹²

$$v = v_- e^- + v_z z. \quad (4.22)$$

Since $A = 0$, we have that $a = A + \frac{3}{2} v = \frac{3}{2} (v_- e^- + v_z z)$. Moreover, from (3.34) and (4.3), we see that v_- is actually constant. Our (4.8) now become

¹² As pointed out after (4.3), the component v_z is immaterial, and can be used to set $d * v = 0$.

$$dz = 2iv_-w \wedge \bar{w}, \quad (4.23a)$$

$$d\omega = 3iv_-e^- \wedge \omega. \quad (4.23b)$$

Moreover, from (B.7) and the fact that $\text{Re}(\sigma \cdot \bar{w}) = 0$ as a consequence of (4.23a), we also have

$$e^- \wedge dw \wedge \bar{w} = \frac{1}{8}(\iota_w \iota_{\bar{w}} de^-)e^- \wedge z \wedge w \wedge \bar{w}. \quad (4.24)$$

A metric of the form (3.7), such that (4.23) and (4.24) hold, is called a *Fefferman metric* [37]. It has the property that, if one rescales the one-form $z \rightarrow \tilde{z} = e^{2\lambda}z$, where λ is a function on the CR manifold M_3 , and one computes new \bar{w} , \tilde{e}^- so that (4.23) and (4.24) are still satisfied, the new metric $\tilde{z}\tilde{e}^- + \tilde{w}\tilde{\bar{w}}$ is equal to $e^{2\lambda}(ze^- + w\bar{w})$.¹³ Notice that (4.23) are very similar to the conditions for Sasaki–Einstein manifolds (which have Euclidean signature rather than Lorentzian, and odd dimension rather than even). The fact that we found a Fefferman metric in the $A = 0$, $z \wedge dz \neq 0$ case is in agreement with the classification in [29].

Let us now consider the case $z \wedge dz = 0$. Using (4.19) and (4.20) it follows that $A^\perp = 0$ implies

$$d_2(\sqrt{H}w) = 0. \quad (4.25)$$

Hence, we can choose a complex coordinate ζ and a function α so that locally $\sqrt{H}w = d\zeta + \alpha du$. We can then rearrange (4.18) as $ds^2 = H^{-1}[du(2dy + d\zeta\bar{\alpha} + d\bar{\zeta}\alpha + \mathcal{F}du) + d\zeta d\bar{\zeta}]$, after suitably redefining \mathcal{F} . Moreover, from (4.21) we learn that the component of $d\alpha$ along w is real. This implies in turn that the one-form $d\zeta\bar{\alpha} + d\bar{\zeta}\alpha$ is closed, up to terms $du \wedge (\dots)$; locally, we can then write $d\zeta\bar{\alpha} + d\bar{\zeta}\alpha = df + gdu$, for some functions f and g . We can now further redefine y and \mathcal{F} to obtain

$$ds^2 = H^{-1}[du(2dy + \mathcal{F}du) + d\zeta d\bar{\zeta}], \quad (4.26)$$

which agrees (locally) with the classification in [29, Eq. (41)].

5. Boundary Geometry from the Bulk

The general analysis of the supersymmetry conditions in the minimal gauged supergravity in five dimensions was performed in [9]. Here we would like to asymptotically expand these results, to extract a set of conditions on a four-dimensional boundary geometry. Not surprisingly, at leading order we find agreement with the conditions that we derived from the CKS equation on the boundary in Sects. 3 and 4.

5.1. Asymptotic expansion of the Bilinears. The analysis in [9] uses the following set of five-dimensional bilinears:

$$\begin{aligned} f\epsilon^{IJ} &= i\bar{\epsilon}^I\epsilon^J, \\ V_\alpha\epsilon^{IJ} &= \bar{\epsilon}^I\gamma_\alpha\epsilon^J, \\ X_{\alpha\beta}^{(1)} + iX_{\alpha\beta}^{(2)} &= -i\bar{\epsilon}^1\gamma_{\alpha\beta}\epsilon^1 = -(i\bar{\epsilon}^2\gamma_{\alpha\beta}\epsilon^2)^*, \\ X_{\alpha\beta}^{(3)} &= \bar{\epsilon}^1\gamma_{\alpha\beta}\epsilon^2 = \bar{\epsilon}^2\gamma_{\alpha\beta}\epsilon^1. \end{aligned} \quad (5.1)$$

¹³ Such a characterization of Fefferman metric can be found in [38]. The metric is defined there by [38, (3.7)]. The term L_θ in that equation is defined composing (4.23a) with the complex structure associated to $T_{1,0}$, the one-dimensional complex subbundle of TM_3 defining the CR structure on M_3 , and corresponds to our term $w\bar{w}$ in (3.7). Moreover, the term $2\theta\sigma$ in [38, (3.7)] is identified with the term e^-z in our (3.7), once we compare our (4.23b) with [38, (3.5)]. Finally, [38, (3.6)] is (4.24).

Here, f is a real scalar, V is a real one-form, and $X^{(i)}$, $i = 1, 2, 3$, are real two-forms¹⁴. We will also define $\Omega = X^{(2)} + iX^{(3)}$.

We can expand the bulk bilinears (5.1) near the boundary using (2.8). In order to facilitate the comparison with the boundary results, let us again define a complex one-form v via the covariant derivative of ϵ as in (4.2), which when plugged into (2.11) yields

$$\eta = i\gamma^\mu \text{Re}(v_\mu \epsilon_+) = \frac{1}{2}\gamma^\mu (i\text{Re}v_\mu \epsilon - \text{Im}v_\mu \gamma_5 \epsilon), \quad (5.2)$$

where we have used that the Majorana condition on ϵ implies that $\epsilon_+^* = \epsilon_-$. Recall that v has a complex part given by (4.4). Recall also the definition of the boundary bilinears (3.2), which together with the Hodge dual of z correspond to the four-dimensional bilinears defined by a single chiral spinor ϵ_+ , and determine an \mathbb{R}^2 structure. Note also that using the properties of the 4d gamma matrices, one can check that

$$*\omega = i\omega \quad (5.3)$$

where the Hodge star is four-dimensional and the metric is the boundary metric $g_{\mu\nu}$.

With these definitions, it is straightforward to compute the asymptotic expansion of the bulk bilinears (5.1) at leading order in r , namely

$$f/8 \sim -\text{Re}v \cdot z, \quad (5.4a)$$

$$\ell^{-1}V/8 \sim r^2 z + r^{-1}\text{Im}v \cdot z dr, \quad (5.4b)$$

$$\ell^{-2}X^{(1)}/8 \sim r dr \wedge z - r^2 (\text{Re}\iota_v *z + \text{Im}v \wedge z), \quad (5.4c)$$

$$\ell^{-2}\Omega/8 \sim ir^3\omega + dr \wedge \iota_v\omega. \quad (5.4d)$$

Using (5.3) and the identity $*\iota_v\omega = iv \wedge \omega$, one also finds that at leading order

$$\ell^{-3}\hat{*}\Omega/8 \sim -ir^3v \wedge \omega - r^2 dr \wedge \omega, \quad (5.5)$$

where $\hat{*}$ denotes the five-dimensional Hodge star.

5.2. Differential conditions from the bulk. The conditions for the existence of supersymmetric solutions in the bulk can be written in terms of a set of differential conditions on the bilinears [9]. We will now expand these conditions near the boundary where the metric is given by (2.2) and the gauge field by (2.5). They read

$$df = -\frac{2}{3}i_V F \quad (5.6a)$$

$$\hat{\nabla}_\alpha V_\beta = \ell^{-1}X_{\alpha\beta}^{(1)} + \dots \quad (5.6b)$$

$$\hat{\nabla}_\alpha X_{\beta\gamma}^{(1)} = 2\ell^{-1}\eta_{\alpha[\beta}V_{\gamma]} + \dots \quad (5.6c)$$

$$\hat{\nabla}_\alpha \Omega_{\beta\gamma} = -i\ell^{-1}\left(2\sqrt{3}\hat{A}_\alpha \Omega_{\beta\gamma} + (*\Omega)_{\alpha\beta\gamma}\right) + \dots \quad (5.6d)$$

where we omitted terms containing F , whenever they are manifestly sub-leading in r .¹⁵

¹⁴ They satisfy a set of algebraic relations that can be found in equations (2.8)–(2.12) of [9] (changing the sign of the metric in order to take into account the opposite choice of signature).

¹⁵ Full expressions can be found in eqs. (2.15), (2.17), (2.18), (2.19) in [9].

We can now further expand the bulk differential conditions (5.6) near the boundary. In the computation, we will need the expressions of the Christoffel symbols for the five-dimensional metric with expansion given in (2.2). We have the following identities

$$\hat{\Gamma}_{rr}^{\mu} = \hat{\Gamma}_{\mu r}^r = 0, \quad \hat{\Gamma}_{rr}^r = -\frac{1}{r}, \quad (5.7)$$

as well as the expansions

$$\hat{\Gamma}_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho} + \mathcal{O}(r^{-1}), \quad \hat{\Gamma}_{\mu r}^{\nu} = \frac{1}{r} \delta_{\mu}^{\nu} + \mathcal{O}(r^{-1}), \quad \hat{\Gamma}_{\mu\nu}^r = -r^3 g_{\mu\nu} + \mathcal{O}(r^2), \quad (5.8)$$

where $\Gamma_{\mu\nu}^{\rho}$ denotes the Christoffel symbols of the four-dimensional metric $g_{\mu\nu}$. Let us start with Eq. (5.6b). Its symmetric part is simply

$$\hat{\nabla}_{(\alpha} V_{\beta)} = 0, \quad (5.9)$$

which states that V^{α} is a Killing vector in the bulk. It is an easy check to see that, at leading order in r , this just says that z^{μ} is a conformal Killing vector on the boundary. To this end, note that the equations having components along r do not give rise to any conditions on the boundary, while the ones without leg along r imply

$$\nabla_{(\mu} z_{\nu)} = -g_{\mu\nu} \text{Im } v \cdot z \equiv \frac{\lambda}{2} g_{\mu\nu}. \quad (5.10)$$

This is the same condition that we found from the purely four-dimensional analysis in (3.18) and (4.4).

Having reproduced the existence of a boundary conformal Killing vector from the gravity analysis, let us now consider the other differential conditions. We have reformulated the conditions on the boundary geometry in (B.8) and (B.9) in such a way to make the comparison with the bulk analysis of this section most straightforward. Plugging (5.4b) and (5.4c) into the anti-symmetric part of (5.6b), we find that again the only non trivial information at leading order in r comes from the four-dimensional part. We get the condition

$$dz = -2(\text{Re } \iota_v * z + \text{Im } v \wedge z), \quad (5.11)$$

which is just the anti-symmetric part of (B.8). Next on the list are Eqs. (5.6c) and (5.6d). At leading order, (5.6c) and the $(\mu\nu 5)$ -part of (5.6d) do not give any new information. On the other hand, upon using (5.5), the four-dimensional part of equation (5.6d) yields

$$(\nabla_{\rho} - 2i A_{\rho}) \omega_{\mu\nu} = i(v \wedge \omega)_{\mu\nu\rho} + i(g_{\rho\nu} \omega_{\mu\sigma} - g_{\rho\mu} \omega_{\nu\sigma}) v^{\sigma}, \quad (5.12)$$

which is precisely the Eq. (B.9).

The final Eq. (5.6a) would seem to be more problematic. It involves the scalar bilinear in the bulk which has no correspondence in the boundary and involves the curvature of the gauge field which we usually neglected because sub-leading in r . However, Eq. (5.6a) expands to

$$df = -\frac{16}{3} \iota_z F, \quad (5.13)$$

a relation that we also found on the boundary. It corresponds indeed to (3.34), as we can see by using (4.3) and (5.4a).

We have thus shown that, as expected, all the conditions for supersymmetry in the bulk reduce to conditions that can be derived from the CKS equation on the boundary. In other words, any supersymmetric bulk solution that can be written asymptotically in the Fefferman–Graham form (2.2) and with a gauge field A satisfying (2.5) reduces to the boundary to a metric with a null conformal Killing vector. This vector is associated with a conformal Killing spinor ϵ_+ charged under A . Vice versa, any Lorentzian metric with a null conformal Killing vector gives rise to a bulk metric (2.2) that solves, at leading order, the supersymmetry conditions of gravity. In this regard, we expect to be able to find a supersymmetric bulk solution with a given boundary condition order by order in r , in the spirit of the Fefferman–Graham construction. It is then a very hard problem to determine which boundary metrics give rise to regular solutions in the bulk. Few examples are known in the literature and they will be reviewed in the next section.

6. Time-Like and Null Solutions in the Bulk

In this section we will analyse in more detail the classification of supersymmetric solutions of minimal five-dimensional gauged supergravity given in [9]. We will demonstrate how to extract the boundary data from a bulk solution and we will discuss how the examples found in [9] fit in the general discussion of supersymmetric boundary geometries.

Let us analyse some general features of the bulk solutions that can be written asymptotically in the Fefferman–Graham form (2.2) and with a gauge field A satisfying (2.5). As discussed in the previous section, the five-dimensional vector V is Killing and its asymptotic expansion (written here as the dual one-form),

$$V \sim r^2 z + r^{-1} \text{Im } v \cdot z \, dr + \dots, \quad \text{Im } v \cdot z = -\frac{1}{2}\lambda, \quad (6.1)$$

gives rise to a null conformal Killing vector z on the boundary. As in [9], we can introduce a coordinate y such that

$$V = \frac{\partial}{\partial y}. \quad (6.2)$$

In this (particularly natural) coordinate system the metric is independent of y and so will be the boundary metric. This means that z is actually Killing and we can identify the bulk coordinate y here with the coordinate y introduced in Sect. 4.2. We also learn that the term $\text{Im } v \cdot z = -\frac{1}{2}\lambda$, which controls the failure of z at being Killing (recall (4.4)), must vanish and

$$V \sim r^2 z + \dots. \quad (6.3)$$

There is no loss of generality here. As discussed in Sect. 4, one can always make z Killing by a Weyl rescaling of the boundary metric. But Weyl rescalings in the boundary are part of diffeomorphisms in the bulk and they can be arranged with a suitable choice of coordinates. In a different coordinate system, for example with an (unnatural) choice of radial coordinate depending on y , we would find that z restricts to a conformal Killing vector on the boundary.

The boundary data can be easily extracted from the bulk metric. It follows from our discussion that the natural framework of where to discuss the boundary supersymmetry is that of the new minimal equation. The boundary metric and gauge field A can be read off from Eqs. (2.2) and (2.5). To have full information about the supersymmetry realised on the boundary we also need the vector v . This is real and satisfies Eq. (4.8a). It can easily be computed starting from z , using for example (4.9); the component of v along z is ambiguous and is determined by requiring (4.6). We will see explicit examples of this procedure in the following.

Note that while z is always null with respect to the boundary metric, the five-dimensional Killing vector V can be null or time-like [9]. This follows from the algebraic constraint (equation (2.8) in [9])

$$V^2 = -f^2. \quad (6.4)$$

For $f \neq 0$, V is time-like while, for $f = 0$, V is null. The time-like and null solutions have different properties and, following [9], we will discuss them separately. Recall from (5.4a) that

$$f \sim -8v \cdot z, \quad (6.5)$$

so the time-like and null bulk solutions correspond to $v \cdot z \neq 0$ and $v \cdot z = 0$, respectively. As already noticed in Sect. 4.3, these correspond to $z \wedge dz \neq 0$ and $z \wedge dz = 0$, respectively. It then follows that the null bulk case corresponds to the non-twisting geometries discussed in Sect. 4.3. In the following, we consider these cases in turn, also discussing two explicit examples as an illustration of our general results.

6.1. Time-like case. In the time-like case, the bulk metric can be written as a time-like fibration over a four-dimensional base B_4 , as

$$ds^2 = -f^2(dy + \tau)^2 + f^{-1}ds^2(B_4), \quad (6.6)$$

where f , τ and $ds^2(B_4)$ do not depend on y . As a one-form, V reads

$$V = -f^2(dy + \tau). \quad (6.7)$$

Supersymmetry in the bulk requires the base B_4 to be Kähler [9]; in particular, as shown in [9], this is equivalent to the equations¹⁶

$$dX^{(1)} = 0, \quad d\Omega = -i\ell^{-1}(2\sqrt{3}\hat{A} - 3f^{-1}V) \wedge \Omega. \quad (6.8)$$

We are interested in metrics that can be written in the Fefferman–Graham form (2.2). As already discussed, such metrics have the properties that at large r , f is independent of r and $V \sim r^2 z$. It then follows that $\tau = O(r^2)$. The boundary supersymmetry is determined by the background fields a and v . v is extracted from (4.8a), while $a = A + \frac{3}{2}v$, where A can be read off from (2.5). Once again, we can check that v is real in such solutions. Indeed, a non-vanishing term $r^{-1}\text{Im } v \cdot z dr$ in (6.1) would contradict the mutual consistency of the two metrics (6.6) and (2.2) by introducing $dydr$ terms.

It is interesting to write explicitly the asymptotic Kähler structure $(X^{(1)}, \Omega)$ on the base manifold B_4 . Using the freedom to take $v = v_- e^- + v_z z$, combining (5.4) with (6.8) we get

$$\ell^{-2}X^{(1)} \sim r dr \wedge z - 2v_- r^2 w \wedge \bar{w} = \frac{1}{2}d\left(r^2 z\right), \quad (6.9)$$

$$\ell^{-2}\Omega \sim (2v_- dr + i r^3 z) \wedge w, \quad (6.10)$$

and correspondingly the Kähler metric reads

$$\ell^{-2}ds^2(B_4) \sim 2v_- \left(\frac{dr^2}{r^2} + r^2 w \bar{w} \right) + \frac{1}{2v_-} r^4 z^2. \quad (6.11)$$

¹⁶ These equations are valid in the gauge $\iota_V \hat{A} = -\frac{\sqrt{3}}{2}f$, used in [9]. (6.8) are therefore equations on the base B_4 .

Eq. (6.9) characterises Kähler cones, however the asymptotic metric is not homogeneous in r , and this is reflected by the $(2, 0)$ -form Ω . Equations (6.9)–(6.11) may be thought of as boundary conditions that a Kähler base B_4 should satisfy. We also note that on surfaces of constant r , Ω pulls back to a form proportional to $z \wedge w$, which characterizes the CR structure on M_3 , as we saw in Sect. 4.2.

In [9, 11], an explicit time-like solution was presented, in which AdS_5 is deformed by a gauge field, and two supercharges are preserved. The Kähler base of the five-dimensional spacetime is the Bergmann space, which is an analytic continuation of \mathbb{CP}^2 . The five-dimensional metric takes the asymptotic form (2.2), with boundary metric

$$ds^2 = -\frac{1}{\ell} \left(dt + \mu \ell^2 \sigma_1 \right) \sigma_3 + \frac{1}{4} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \quad (6.12)$$

where the σ 's are right-invariant one-forms on S^3 :

$$\begin{aligned} \sigma_1 &= \sin \phi d\theta - \sin \theta \cos \phi d\psi, \\ \sigma_2 &= \cos \phi d\theta + \sin \theta \sin \phi d\psi, \\ \sigma_3 &= d\phi + \cos \theta d\psi. \end{aligned} \quad (6.13)$$

This is a non-Einstein, non-conformally flat metric on $\mathbb{R} \times S^3$. In our conventions, the gauge field at the boundary reads

$$A = \frac{3}{2\ell} (dt + \mu \ell^2 \sigma_1). \quad (6.14)$$

Here, μ is a parameter of the solution. When $\mu = 0$, the gauge field is trivial, the boundary metric becomes the standard one on $\mathbb{R} \times S^3$ (after a coordinate transformation), and the bulk spacetime is just AdS_5 .

Identifying the frame as

$$e^+ = \frac{\sigma_3}{2}, \quad e^- = -\frac{2}{\ell} (dt + \mu \ell^2 \sigma_1) + \frac{\sigma_3}{2}, \quad w = \frac{1}{2} (\sigma_1 - i\sigma_2), \quad (6.15)$$

we see that (6.12) agrees with our general description of Sect. 4.2, with the coordinate identification $\{y, u, x^1, x^2\} = \{-t/\ell, \phi, \theta, \psi\}$. We also need to identify

$$H = 2, \quad \mathcal{F} = \frac{1}{4}, \quad \beta = \cos \theta d\psi, \quad \varrho = -\mu \ell \sigma_1, \quad (6.16)$$

and the metric h_{mn} with the round metric on S^2 . One can also check that the gauge field in (6.14) is consistent with our general formulae in Sect. 4.2. Using (4.14), we find that $v^\perp = \frac{1}{2} e^-$ and this can be completed by choosing $e^- \cdot v = -1$, so that

$$v = \frac{1}{2} (e^- - e^+) = -e^0 \quad (6.17)$$

satisfies $d * v = 0$. Finally, $a = A + \frac{3}{2} v = 0$ is consistent with (4.15) and (4.16).

We checked that with these values of v and a , the new minimal equation (4.1) is solved by a constant spinor ϵ_+ satisfying the projection $\gamma^0 \gamma^1 \epsilon_+ = \epsilon_+$. This shows that the background preserves precisely two supercharges.¹⁷ Finally, we note that from the

¹⁷ Note that in order to map the frame chosen in [11] into the five-dimensional frame used here, one needs to perform an r -dependent Lorentz transformation. Acting on the spinors, this transforms the spinors in [11], which are independent of r , into r -dependent spinors, with asymptotic form given in (2.8). Note also that the r -dependence of the spinors in [11] arises as a consequence of a different gauge for A . In particular, in [11]: $A \cdot z = 0$.

point of view of the boundary geometry we could deform the metric on S^3 in various ways. However, which deformations can be completed to a non-singular solution in the bulk is a very hard question to address.

6.2. Null case. In the null case, $f = 0$ and the bulk metric can be written as [9]

$$ds^2 = -2\hat{H}^{-1}du \left(dy + \frac{1}{2}\mathcal{F}du \right) + \hat{H}^2\gamma_{mn}dx^m dx^n, \quad (6.18)$$

where \hat{H} , γ_{mn} and \mathcal{F} depend only on u and x^m , $m = 1, 2, 3$, but not on y . Here

$$V = \hat{H}^{-1}du \quad (6.19)$$

and by comparison with (6.3) we see that $\hat{H}^{-1} = r^2 H^{-1} + \dots$, and $z = H^{-1}du$, in agreement with the results of Sect. 4.3.

Explicit asymptotically locally AdS solutions in the null case are also discussed in [9]. These are the magnetic string solutions of [39, 40]. The boundary is $\mathbb{R}^{1,1} \times M_2$, with metric (after some obvious rescaling)

$$ds^2 = 2dudy + ds^2(M_2), \quad (6.20)$$

and the gauge field is

$$F = -\frac{k}{2}\text{vol}(M_2). \quad (6.21)$$

Here, M_2 is S^2 if $k > 0$ (with radius $k^{-1/2}$), \mathbb{T}^2 if $k = 0$, or the hyperbolic space \mathbb{H}^2 if $k < 0$ (with radius $(-k)^{-1/2}$). The bulk space-time has a regular horizon when $k < 0$, while it has a naked singularity when $k > 0$. Setting $H = 1$, $\mathcal{F} = 0$, we find that the formulae in our Sect. 4.3 are consistent with $v = 0$ and $F = da$.

Notice that these bulk solutions can be easily Wick-rotated to Euclidean signature, giving boundary metrics on $\mathbb{R}^2 \times M_2$. In the case $M_2 = \mathbb{H}^2$, the Wick-rotated bulk solution is non-singular, and interpolates between Euclidean AdS₅ asymptotically and $\mathbb{H}^3 \times \mathbb{H}^2$ in the interior. The similar case $\mathbb{T}^2 \times M_2$ was discussed in [41].

7. Discussion

Motivated by the recent successful applications of localization techniques in the context of supersymmetric gauge theories on Euclidean curved manifolds, in this paper we have studied four-dimensional rigid supersymmetry on curved backgrounds in Lorentzian signature. We have shown that the backgrounds are quite different in the two cases, and in general they are not (and can not) be simply related by a Wick rotation. In Euclidean signature, preserved supersymmetry for a theory with an R-symmetry leads to a charged conformal Killing spinor equation, and is equivalent to the four-dimensional manifold being complex [12, 13]. Here we demonstrated that in Lorentzian signature, solutions to the same equation are characterized by the existence of a conformal Killing vector on the four-dimensional manifold. We have also discussed how rigid supersymmetry arises on the boundary of supersymmetric asymptotically locally AdS solutions of minimal five-dimensional gauged supergravity, which were analysed previously [9]. It would be very interesting to perform a similar comparison between Euclidean rigid supersymmetry on the boundary and Euclidean five-dimensional supergravity solutions, and we plan to

address this in future work. Here we have illustrated the relationship between Lorentzian supersymmetry in the bulk and on the boundary in some examples [9, 29], where the boundary metric is that of a (non-conformally flat) $\mathbb{R} \times S^3$ or $\mathbb{R}^{1,1} \times M_2$. It would be interesting to construct new examples of non-singular supergravity solutions with other conformal boundary metrics and study their field theory duals.

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A. Spinor Conventions

In this appendix we collect our spinor conventions. The Clifford(1, 4) gamma matrices γ^α satisfy

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}, \quad \gamma_\alpha^\dagger = \gamma_0 \gamma_\alpha \gamma_0, \quad \gamma_\alpha^t = C \gamma_\alpha C^{-1}, \quad (\text{A.1})$$

where the five-dimensional charge conjugation matrix C satisfies $C = -C^t = C^* = -C^{-1}$. We adopt a representation of the Clifford algebra in which the first four gamma matrices are real, while $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is purely imaginary. Then $\gamma^1, \gamma^2, \gamma^3$ are symmetric, while γ^0 and γ^5 are anti-symmetric. In this case, a consistent choice of the charge conjugation matrix is $C = i\gamma^0\gamma^5$. Our spinors are commuting. Furthermore, for five-dimensional spinors, the symplectic-Majorana condition is

$$\bar{\epsilon}^I = (\epsilon^I)^t C, \quad (\text{A.2})$$

where ϵ^I , $I = 1, 2$, are Dirac spinors and we define $\bar{\epsilon}^I = \epsilon^{IJ} \epsilon^{J\dagger} \gamma^0$, with ϵ^{IJ} being the antisymmetric symbol, such that $\epsilon^{12} = +1$. So a symplectic-Majorana pair ϵ^I carries in total eight real degrees of freedom.

B. Intrinsic Torsions and Differential Forms

In this appendix we will explain how to obtain the system (3.13), which allows to compute the intrinsic torsions p and q by using differential forms and exterior differentials only, and not spinors. We will also give explicit expressions for the differentials and covariant derivatives of the vielbein $\{z, e^-, w, \bar{w}\}$ and the two form ω corresponding to a conformal Killing spinor.

We start with the derivation of system (3.13), consisting of the derivatives of the elements of the vielbein. The easiest to compute is dz , (3.13a). z is a spinor bilinear, as can be seen in (3.1); so its derivative can be computed in the standard way. We actually even gave its covariant derivative in (3.17); indeed by antisymmetrizing its μ and ν indices one obtains (3.13a).

dw and de^- are trickier because w and e^- are not directly expressed as bilinears of ϵ_+ ; as explained in Sect. 3.1, they are an additional piece of data, subject to the ambiguity (3.8). The two-form ω , on the other hand, is a bispinor, defined in (3.1), and one can compute $d\omega$ again in a standard way; one gets (3.14b). Now, since w is that $\omega = z \wedge w$, from $d(z \wedge w) = z \wedge dw - dz \wedge w$ one sees that

$$z \wedge dw = -2i \text{Imp} \wedge z \wedge w - 2q \wedge e^- \wedge z; \quad (\text{B.1})$$

from this, it follows that one can write dw as in (3.13b), for some one-form ρ .

We can give an alternative characterization of ρ by computing dw in a different way: namely, by writing

$$\bar{\epsilon}_- \gamma_\mu e^- \epsilon_+ = -4(e^-)^v \omega_{\mu v} = 8w_\mu \quad (\text{B.2})$$

and deriving the left hand side. For this, we need to compute

$$\begin{aligned} D_\mu(e^- \epsilon_+) &= [D_\mu, e^-] \epsilon_+ + e^- (p_\mu \epsilon_+ + q_\mu e^- \epsilon_-) = (D_\mu e_v^- + p_\mu e_v^-) \gamma^v \epsilon_+ \\ &= -w^\mu D_\mu e_v^- \epsilon_- + \left(p_\mu + \frac{1}{2} z^v D_\mu e_v^- \right) e^- \epsilon_+. \end{aligned} \quad (\text{B.3})$$

Here we have used the definition (3.12) of the intrinsic torsions, and (3.11). Using this and (B.2), we get again (3.13b), where now we see that

$$\rho_\mu = \frac{1}{4} w^v D_\mu e_v^-, \quad \text{Re } p_\mu = -\frac{1}{4} z^v D_\mu e_v^-. \quad (\text{B.4})$$

This now suggests a way of computing e^- . Using

$$e_\mu^- = \frac{1}{16} \bar{\epsilon}_+ e^- \gamma_\mu e^- \epsilon_+, \quad (\text{B.5})$$

the expression for $D_\mu(e^- \epsilon_+)$ computed in (B.3), and the formula for ρ in (B.4), we obtain (3.13c).

The system of Eq. (3.13) is general and applies to any vielbein constructed from a chiral fermion ϵ_+ as explained in Sect. 3.1; ϵ_+ is not assumed to satisfy any particular differential equation. It is of some interest to go on-shell and write the derivatives of the elements of a vielbein corresponding to a solution of the CKS equation (3.15). Imposing the constraints (3.16) on the torsions we have

$$\begin{aligned} dz &= 2\text{Re}(q \cdot \bar{w}) e^- \wedge z + 2\text{Im}(q \cdot \bar{w}) i w \wedge \bar{w} + 4\text{Re}((q \cdot e^-) z \wedge \bar{w}), \\ dw &= (2iA + (\text{Re}(q \cdot \bar{w}) + 3i\text{Im}(q \cdot \bar{w})) e^- - (q \cdot e^-) \bar{w}) \wedge w - 2\sigma \wedge z, \\ de^- &= 4\text{Re}(\sigma \wedge \bar{w}), \end{aligned} \quad (\text{B.6})$$

where $\sigma = \rho - \frac{1}{2}(q \cdot e^-) e^-$. Equation (3.21) easily follows from these equations. By construction, the set of Eq. (B.6) implies that z is conformal Killing. These equations are also interesting because they can be used to determine the gauge field A .

In the new minimal case, using (4.3) and the definition $a = A + \frac{3}{2}v$ we find

$$\begin{aligned} dz &= i \iota_v(z \wedge w \wedge \bar{w}), \\ dw &= 2i \left(a - \frac{3}{4}(v \cdot e^-) z - \frac{1}{2}(v \cdot w) \bar{w} \right) \wedge w - 2\sigma \wedge z, \\ de^- &= 4\text{Re}(\sigma \wedge \bar{w}), \end{aligned} \quad (\text{B.7})$$

from which Eq. (4.8) follows. The set of equations (B.7) implies that z is Killing. They allow to determine uniquely the background fields a and v .

Finally, we also give some expressions for the covariant derivatives of the forms z and ω corresponding to a solution of the CKS equation, which have been used in the bulk to boundary comparison in Sect. 5. The expressions are not particularly nice in terms of the torsions p and q but become simple if we replace q with the vector v using (4.3). This formal redefinition can be used both in the case of solutions of the CKS equation and in the case of solutions of the new minimal conditions. As discussed in Sect. 4, the only

difference between the two cases is that, for conformal Killing spinors, v has a complex part given by (4.4). We also use $a = A + \frac{3}{2}v$. By explicitly differentiating the bilinears z and ω and using (3.16), we find

$$\nabla_\nu z_\mu = 2 \operatorname{Im} v_{[\mu} z_{\nu]} + \operatorname{Re} v^\tau (*z)_{\mu\nu\tau} - g_{\mu\nu} z_\tau \operatorname{Im} v^\tau, \quad (\text{B.8})$$

$$\nabla_\tau \omega_{\mu\nu} = 2i A_\tau \omega_{\mu\nu} + i(v \wedge \omega)_{\mu\nu\tau} + i(g_{\nu\tau} v^\sigma \omega_{\mu\sigma} - g_{\mu\tau} v^\sigma \omega_{\nu\sigma}). \quad (\text{B.9})$$

As expected, by symmetrizing and anti-symmetrizing and using (4.3) and (4.4) we recover known formulae: (3.18), the first expression in (B.6) and (4.8).

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